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The geometry of $D = 11$ null Killing spinors

Jerome P. Gauntlett*

Perimeter Institute for Theoretical Physics
Waterloo, ON, N2J 2W9, Canada
E-mail: jgauntlett@perimeterinstitute.ca

Jan B. Gutowski

Mathematical Institute
Oxford, OX1 3LB, U.K.
E-mail: gutowski@maths.ox.ac.uk

Stathis Pakis

Blackett Laboratory, Imperial College
London, SW7 2BZ, U.K.
E-mail: s.pakis@imperial.ac.uk

Abstract: We determine the necessary and sufficient conditions on the metric and the four-form for the most general bosonic supersymmetric configurations of $D = 11$ supergravity which admit a null Killing spinor i.e. a Killing spinor which can be used to construct a null Killing vector. This class covers all supersymmetric time-dependent configurations and completes the classification of the most general supersymmetric configurations initiated in [1].

Keywords: M-Theory, Supergravity Models.

*On leave from: Blackett Laboratory, Imperial College, London, SW7 2BZ, U.K.

Contents

1. Introduction

Supersymmetric bosonic solutions of supergravity theories have been important in many developments in string/M-theory. Recently there has been significant progress in determining the most general kinds of geometries that underly all such solutions of a particular theory [1]–[5]. In addition to providing a deeper understanding of known classes of solutions, this analysis is useful in precisely characterising geometries of interest when explicit solutions are difficult to come by. Thus, for example, a uniqueness theorem for a class of supersymmetric black holes was found in [6]. This analysis also provides new techniques for constructing explicit solutions that have been hitherto missed by other approaches based on guessing ansatzë.

The basic idea is to translate the condition for supersymmetry, the existence of a Killing spinor, into differential conditions on the tensors that can be constructed as bi-linears from the Killing spinor. This approach was first employed by Tod some time ago [7, 8], following [9], to analyse N=2 supergravity in $D = 4$. For the case of the minimal supergravity theory, using features specific to $D = 4$, it was possible to explicitly construct the most general supersymmetric solutions. In higher dimensions, it is not possible to find all of the solutions in closed form but nevertheless, a precise description of the geometry can be made.

One of the most interesting supergravity theories to study is $D = 11$ supergravity as it describes the low-energy limit of M-theory. In [1], a programme for classifying all supersymmetric solutions of $D = 11$ supergravity was outlined based on the above strategy. A key observation of [1] is that in organising the calculations it is very useful to note that the Killing spinors define a privileged G-structure. That is, in this case, a reduction of the $\mathrm{Spin}(10, 1)$ frame bundle to a G-sub-bundle (for some general mathematical discussion on G-structures see e.g. [10]). The utility of G-structures in analysing restricted classes of supergravity solutions has also been shown in [11]–[22] (see also [23, 24]).

One general feature of the results of [1] is that generalised calibrations [25]–[27] naturally emerge from the conditions for supersymmetry. This was noticed earlier for a restricted class of configurations of D=10 supergravity in [12], and is related to the fact that supersymmetric geometries with non-vanishing fluxes arise when branes wrap calibrated cycles in special holonomy manifolds, after taking the back reaction into account. It was shown in [28] that the connections with generalised calibrations found in [1] lead to a simple proposal for the topological charges appearing in the supersymmetry algebras of membranes and fivebranes propagating in general supersymmetric backgrounds.

The most general supersymmetric solutions of $D = 11$ supergravity preserve at least one supersymmetry and hence admit at least one Killing spinor. However, there are two kinds of spinors of $\mathrm{Spin}(10, 1)$ which are distinguished by whether the vector that can be constructed as a bi-linear from the spinor is time-like or null [30]. Moreover, the vector constructed from the Killing spinor is necessarily Killing. Consequently there are two kinds of supersymmetric solutions of $D = 11$ supergravity: those admitting a "time-like Killing spinor", which can be used to construct a time-like Killing vector, and those admitting a "null Killing spinor", which can be used to construct a null Killing vector.[1]

Supersymmetric solutions with a single time-like Killing spinor were analysed in detail in [1]. It was shown that the Killing spinor defines a preferred SU(5)-structure and this was used to determine the most general local form of the metric and the four-form field strength. The focus of this paper will be to perform a similar analysis for supersymmetric solutions

[1]Note that geometries preserving more than one supersymmetry can be in both classes. Also note, for example, that it is a priori possible for a geometry preserving just one supersymmetry to be in the null class and also admit a time-like Killing vector, but the time-like Killing vector will not be built from the Killing spinor.

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admitting a null Killing-spinor. These solutions have a $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, which can be used to determine the general local form of the solutions. It is worth emphasising that since all of the solutions in the time-like class are stationary, in this sense any time-dependent supersymmetric solutions are necessarily in the null class.

The results of this paper and [1] therefore provide a classification of the most general supersymmetric solutions of $D = 11$ supergravity, preserving at least $1/32$ supersymmetry. This classification can in principle be refined [1] by analysing the additional conditions placed on the geometries preserving more than one supersymmetry. For example, if the geometry preserves two supersymmetries, the two Killing spinors could both be null, both time-like or be one of each. These solutions will thus be special cases of the geometries presented here, or in [1] (or both), satisfying extra constraints. Similarly, solutions preserving more supersymmetries will be further restricted. It will be very interesting to pursue this further. It is noteworthy that the classification of maximally supersymmetric configurations has already been carried out in [29] using methods specific to this case.

The plan of this paper is as follows. In section 2 we evaluate the algebraic and differential identities which the various bi-linears constructed from a null Killing spinor must satisfy. In section 3 we use these expressions to constrain the eleven-dimensional geometry and to fix almost all of the components of the four-form. We also demonstrate that these necessary conditions are in fact sufficient by demonstrating that the configurations always admit a Killing spinor. For the convenience of the reader we have summarised the main results of this section in section 3.3. In section 4 we introduce local co-ordinates on the eleven-dimensional manifold in which the constraints take a rather simple form. In particular, we show that the Spin(7) invariant 4-form must be conformally anti-self-dual. We again summarise the main result of this section in a separate sub-section. We demonstrate in section 5 how the resolved membrane solution of [34], the $1/32$ supersymmetric membrane/wave solution of [35], and the basic fivebrane solution can be obtained from our construction. This provides non-trivial checks on our calculations as well as providing some intuition into the kinds of $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures allowed by supersymmetry. In addition, the formalism allows us to generalise the resolved membrane solution by the addition of a gravitational wave thus combining the solutions of [34, 35]. In section 6 we analyse the special case when the four-form vanishes, recovering the general local-form for the solution found in [30]. Section 7 briefly concludes. The paper finishes with several appendices A–D containing some useful technical information.

2. Killing spinors and differential forms

The bosonic fields of $D = 11$ supergravity consist of a metric, g, and a three-form potential C with four-form field strength $F = dC$. The action for the bosonic fields is given by

$$S = \frac{1}{2\kappa^2} \int d^{11}x\sqrt{-g}R - \frac{1}{2}F \wedge *F - \frac{1}{6}C \wedge F \wedge F\,. \tag{2.1}$$

The equations of motion[2] and the Bianchi identity are thus given by

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{12}\left(F_{\mu\sigma_1\sigma_2\sigma_3}F_\nu{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{12}g_{\mu\nu}F^2\right) &= 0 \\
d * F + \frac{1}{2}F \wedge F &= 0 \\
dF &= 0\,,
\end{aligned}
\tag{2.2}
$$

where $F^2 = F_{\mu_1\mu_2\mu_3\mu_4}F^{\mu_1\mu_2\mu_3\mu_4}$. A solution of these equations preserves at least one supersymmetry if it admits at least one Killing spinor, ϵ, which solves

$$
\nabla_\mu\epsilon + \frac{1}{288}[\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}]F_{\nu_1\nu_2\nu_3\nu_4}\epsilon = 0\,. \tag{2.3}
$$

Our conventions are outlined in appendix A.

Consider a configuration (g, F) that admits a single Killing spinor ϵ. We can then define the following one-, two- and five-forms:

$$
\begin{aligned}
K_\mu &= \bar{\epsilon}\Gamma_\mu\epsilon \\
\Omega_{\mu_1\mu_2} &= \bar{\epsilon}\Gamma_{\mu_1\mu_2}\epsilon \\
\Sigma_{\mu_1\mu_2\mu_3\mu_4\mu_5} &= \bar{\epsilon}\Gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}\epsilon\,.
\end{aligned}
\tag{2.4}
$$

Note that in the above construction we take ϵ to be a commuting spinor. Of course the supersymmetry parameter is an anticommuting spinor but since we are interested in purely bosonic supersymmetric configurations the only relevant supersymmetry variation is that of the gravitino which yields the Killing spinor equation. This is linear in the spinor and hence the existence of a commuting Killing spinor is equivalent to the preservation of a supersymmetry. As noted in [1], using an argument presented in [6], we can assume without loss of generality that ϵ is nowhere vanishing. If there is more than one linearly independent Killing spinor, then additional forms including scalars, three and four-forms can also be defined. However, here we shall only consider the most general case of a single Killing spinor.

These differential forms are not all independent. They satisfy certain algebraic relations which are consequences of the underlying Clifford algebra. The traditional way of obtaining these is by repeated use of Fierz identities and some were presented in [1]. Alternatively we can use the fact that the forms, or equivalently the Killing spinor, give rise to privileged G-structures with $G \subset \mathrm{Spin}(10,1)$. Indeed it was argued in [1], using results of [30], that there are two possibilities. If K is null everywhere then the forms give rise to a globally defined $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure. If K is timelike at a point, on the other hand, then it is timelike in a neighbourhood of this point and the forms then define a privileged SU(5) structure in this neighbourhood. It is not possible to have a spacelike K.

[2] Note that in M-theory, the field-equation for the four-form receives higher order gravitational corrections given, in our conventions, by [1, equation (2.4)]. Since most of our analysis only concerns the Killing spinor equation given here in (2.3), including this correction at the level of the gauge equations of motion is straightforward.

The necessary and sufficient conditions for a configuration (g, F) to admit time-like Killing spinors were analysed in [1]. Here we will focus on the null case. It is therefore convenient to work in a null basis

$$ds^2 = 2e^+e^- + e^ie^i + e^9e^9 \tag{2.5}$$

with $i = 1, \ldots, 8$ and

$$K = e^+ . \tag{2.6}$$

We choose an orientation such that $\epsilon_{+-123456789} = -1$. The most convenient way to determine the forms Ω and Σ for null spinors is to construct them from a specific null spinor. Such a spinor can be fixed by demanding that it satisfies the following projections[3]

$$\begin{aligned}\Gamma_{1234}\epsilon &= \Gamma_{3456}\epsilon = \Gamma_{5678}\epsilon = \Gamma_{1357}\epsilon = -\epsilon \\ \Gamma^+\epsilon &= 0 .\end{aligned} \tag{2.7}$$

Note that these conditions automatically imply that $\Gamma^9\epsilon = \epsilon$. It is then straightforward to deduce that

$$\begin{aligned}\Omega &= e^+ \wedge e^9 \\ \Sigma &= e^+ \wedge \phi ,\end{aligned} \tag{2.8}$$

where $\phi = \frac{1}{4!}\phi_{i_1i_2i_3i_4}e^{i_1i_2i_3i_4}$ is the Spin(7) invariant four-form whose only non-vanishing components are given by

$$\begin{aligned}-\phi = {} & e^{1234} + e^{1256} + e^{1278} + e^{3456} + e^{3478} + e^{5678} + \\ & + e^{1357} - e^{1368} - e^{1458} - e^{1467} - e^{2358} - e^{2367} - e^{2457} + e^{2468} .\end{aligned} \tag{2.9}$$

Observe that the action of $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \ltimes \mathbb{R}$ on the basis 1-forms is given by

$$\begin{aligned}e^+ &\to (e^+)' = e^+ \\ e^- &\to (e^-)' = e^- - \frac{1}{2}(\alpha^2 + p_ip^i)e^+ - \alpha e^9 - Q_{ij}p^ie^j \\ e^9 &\to (e^9)' = e^9 + \alpha e^+ \\ e^i &\to (e^i)' = Q^i{}_je^j + p^ie^+ ,\end{aligned} \tag{2.10}$$

where $Q \in \mathrm{Spin}(7)$, and $p_i = \delta_{ij}p^j$; in particular, we see that these transformations not only preserve the metric but also K, Ω and Σ.

In appendix B we present an alternative derivation of these results using Fierz identities; in particular this provides an independent check of some of the results of [30].

[3] From a physical point of view these projections are equivalent to those arising when a fivebrane wraps a Cayley four-cycle [31]. Corresponding supergravity solutions were presented in [32].

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The covariant derivatives of the differential forms were calculated in [1]. The result for both the timelike and the null case is:

$$\nabla_\mu K_\nu = \frac{1}{6}\Omega^{\sigma_1\sigma_2}F_{\sigma_1\sigma_2\mu\nu} + \frac{1}{6!}\Sigma^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} * F_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\mu\nu} \tag{2.11}$$

$$\begin{aligned}\nabla_\mu\Omega_{\nu_1\nu_2} &= \frac{1}{3.4!}g_{\mu[\nu_1}\Sigma_{\nu_2]}{}^{\sigma_1\sigma_2\sigma_3\sigma_4}F_{\sigma_1\sigma_2\sigma_3\sigma_4} + \frac{1}{3.3!}\Sigma_{\nu_1\nu_2}{}^{\sigma_1\sigma_2\sigma_3}F_{\mu\sigma_1\sigma_2\sigma_3} - \\ &\quad - \frac{1}{3.3!}\Sigma_{\mu[\nu_1}{}^{\sigma_1\sigma_2\sigma_3}F_{\nu_2]\sigma_1\sigma_2\sigma_3} + \frac{1}{3}K^\sigma F_{\sigma\mu\nu_1\nu_2}\end{aligned} \tag{2.12}$$

$$\begin{aligned}\nabla_\mu\Sigma_{\nu_1\nu_2\nu_3\nu_4\nu_5} &= \frac{1}{6}K^\sigma * F_{\sigma\mu\nu_1\nu_2\nu_3\nu_4\nu_5} - \frac{10}{3}F_{\mu[\nu_1\nu_2\nu_3}\Omega_{\nu_4\nu_5]} - \frac{5}{6}F_{[\nu_1\nu_2\nu_3\nu_4}\Omega_{\nu_5]\mu} - \\ &\quad - \frac{10}{3}g_{\mu[\nu_1}\Omega_{\nu_2}{}^{\sigma}F_{|\sigma|\nu_3\nu_4\nu_5]} + \frac{5}{6}F_{\mu[\nu_1|\sigma_1\sigma_2|} * \Sigma^{\sigma_1\sigma_2}{}_{\nu_2\nu_3\nu_4\nu_5]} + \\ &\quad + \frac{5}{6}F_{[\nu_1\nu_2|\sigma_1\sigma_2|} * \Sigma^{\sigma_1\sigma_2}{}_{\nu_3\nu_4\nu_5]\mu} - \frac{5}{9}g_{\mu[\nu_1}F_{\nu_2|\sigma_1\sigma_2\sigma_3|} * \Sigma^{\sigma_1\sigma_2\sigma_3}{}_{\nu_3\nu_4\nu_5]}\,.\end{aligned} \tag{2.13}$$

The exterior derivatives of the forms are thus given by

$$dK = \frac{2}{3}i_\Omega F + \frac{1}{3}i_\Sigma * F \tag{2.14}$$

$$d\Omega = i_K F \tag{2.15}$$

$$d\Sigma = i_K * F - \Omega \wedge F\,, \tag{2.16}$$

where e.g. $(i_\Omega F)_{\mu\nu} = (1/2!)\Omega^{\rho_1\rho_2}F_{\rho_1\rho_2\mu\nu}$.

From the first equation in (2.11) we can immediately deduce that K is a Killing vector. Moreover, using the Bianchi identity, it is simple to show that

$$\mathcal{L}_K F = 0\,. \tag{2.17}$$

Thus any geometry (g, F) admitting a Killing spinor possesses a symmetry generated by K. In addition, the Lie-derivatives of Ω and Σ with respect to K also vanish:

$$\begin{aligned}\mathcal{L}_K\Omega &= 0 \\ \mathcal{L}_K\Sigma &= 0\,.\end{aligned} \tag{2.18}$$

3. The geometry of null Killing spinors

Our aim is to extract from the differential conditions (2.11), (2.12) and (2.13) the necessary and sufficient conditions on the geometry and the four-form field strength in order that they admit null Killing spinors. In the next two subsections we derive the necessary conditions and then show that they are sufficient. In the third subsection we have summarised the results. The final brief subsection states the extra conditions required in order that the configuration also solves the equations of motion.

3.1 Necessary conditions

We will need to decompose various forms carrying totally anti-symmetric $SO(8)$ representations into Spin(7) reps. If we denote by Λ^p the space of p-forms constructed from e^i only, we have the following decompositions:

$$\begin{aligned}
\Lambda^1 &: \ \mathbf{8} \rightarrow \ \mathbf{8} \\
\Lambda^2 &: \mathbf{28} \rightarrow \mathbf{21} + \mathbf{7} \\
\Lambda^3 &: \mathbf{56} \rightarrow \mathbf{48} + \mathbf{8} \\
\Lambda^4 &: \mathbf{70} \rightarrow \mathbf{35} + \mathbf{1} + \mathbf{7} + \mathbf{27}\,.
\end{aligned} \tag{3.1}$$

For two-forms the projections can be written explicitly as

$$\begin{aligned}
(P^{\mathbf{21}}\alpha)_{i_1i_2} &= \frac{3}{4}\left(\alpha_{i_1i_2} + \frac{1}{6}\phi_{i_1i_2}{}^{j_1j_2}\alpha_{j_1j_2}\right) \\
\left(P^{\mathbf{7}}\alpha\right)_{i_1i_2} &= \frac{1}{4}\left(\alpha_{i_1i_2} - \frac{1}{2}\phi_{i_1i_2}{}^{j_1j_2}\alpha_{j_1j_2}\right).
\end{aligned} \tag{3.2}$$

For three-forms we have,

$$\begin{aligned}
\left(P^{\mathbf{48}}\alpha\right)_{i_1i_2i_3} &= \frac{6}{7}\left(\alpha_{i_1i_2i_3} + \frac{1}{4}\phi^{j_1j_2}{}_{[i_1i_2}\alpha_{i_3]j_1j_2}\right) \\
\left(P^{\mathbf{8}}\alpha\right)_{i_1i_2i_3} &= \frac{1}{7}\left(\alpha_{i_1i_2i_3} - \frac{3}{2}\phi^{j_1j_2}{}_{[i_1i_2}\alpha_{i_3]j_1j_2}\right).
\end{aligned} \tag{3.3}$$

For four-forms the $\mathbf{35}$ is the anti-self-dual piece, while

$$\begin{aligned}
\left(P^{\mathbf{1}}\alpha\right)_{i_1i_2i_3i_4} &= \frac{1}{336}\phi_{i_1i_2i_3i_4}\phi^{j_1j_2j_3j_4}\alpha_{j_1j_2j_3j_4} \\
\left(P^{\mathbf{7}}\alpha\right)_{i_1i_2i_3i_4} &= \frac{1}{8}\alpha_{i_1i_2i_3i_4} + \frac{1}{224}\phi_{i_1i_2i_3i_4}\phi^{j_1j_2j_3j_4}\alpha_{j_1j_2j_3j_4} - \\
&\quad - \frac{3}{224}\phi^{j_1j_2}{}_{[i_1i_2}\phi_{i_3i_4]}{}^{j_3j_4}\alpha_{j_1j_2j_3j_4} + \frac{5}{168}\phi^{j_1}{}_{[i_1i_2i_3}\phi_{i_4]}{}^{j_2j_3j_4}\alpha_{j_1j_2j_3j_4} \\
\left(P^{\mathbf{27}}\alpha\right)_{i_1i_2i_3i_4} &= \frac{3}{8}\alpha_{i_1i_2i_3i_4} - \frac{1}{224}\phi_{i_1i_2i_3i_4}\phi^{j_1j_2j_3j_4}\alpha_{j_1j_2j_3j_4} + \\
&\quad + \frac{15}{224}\phi^{j_1j_2}{}_{[i_1i_2}\phi_{i_3i_4]}{}^{j_3j_4}\alpha_{j_1j_2j_3j_4} + \frac{1}{56}\phi^{j_1}{}_{[i_1i_2i_3}\phi_{i_4]}{}^{j_2j_3j_4}\alpha_{j_1j_2j_3j_4}\,.
\end{aligned} \tag{3.4}$$

The identities satisfied by ϕ that we used to construct these projections, as well as various identities satisfied by the forms in different representations are presented in appendix B. Note that as far as we know, the projections for the four-forms are new.

By analyzing the expressions (2.14), (2.15) for dK and $d\Omega$ we immediately find that some components of the flux must vanish:

$$\begin{aligned}
F_{-i_1i_2i_3} &= 0 \\
F^{\mathbf{7}}_{-i_1i_29} &= 0\,.
\end{aligned} \tag{3.5}$$

In addition we get

$$\begin{aligned}
dK = de^+ &= \left(-\frac{2}{3}F_{+-i9} + \frac{1}{3}\frac{1}{3!}\phi_i{}^{j_1j_2j_3}F_{j_1j_2j_39}\right)e^{+i} + \\
&\quad + \frac{1}{3}\left(\frac{1}{4!}\phi^{i_1i_2i_3i_4}F_{i_1i_2i_3i_4}\right)e^{+9} + \frac{1}{2}\left(F_{-i_1i_29}\right)e^{i_1i_2}\,.
\end{aligned} \tag{3.6}$$

Note that K satisfies

$$K \wedge dK = \frac{1}{2}(F_{-i_1 i_2 9}) e^{+i_1 i_2} \tag{3.7}$$

which implies that K is not hyper-surface orthogonal in general.[4]

To proceed it is useful to write the constraints on F in terms of the spin connection ω defined by

$$(\nabla_\alpha e^\beta)_\lambda \equiv -\omega_\alpha{}^\beta{}_\lambda \,. \tag{3.8}$$

In particular, as K is Killing, we have

$$\omega_{(\alpha\beta)-} = 0 \tag{3.9}$$

and (3.6) can be rewritten as

$$\begin{aligned}
\omega_{+i-} &= -\frac{1}{3}F_{+-i9} + \frac{1}{36}\phi_i{}^{j_1 j_2 j_3} F^{\mathbf{8}}_{j_1 j_2 j_3 9} \\
\omega_{+9-} &= \frac{1}{144}\phi^{i_1 i_2 i_3 i_4} F^{\mathbf{1}}_{i_1 i_2 i_3 i_4} \\
\omega_{i_1 i_2 -} &= \frac{1}{2}F_{-9 i_1 i_2} \\
\omega_{i9-} &= 0 \,.
\end{aligned} \tag{3.10}$$

Next we examine (2.12). This implies the following additional relationships between the spin connection and the gauge field strength:

$$\begin{aligned}
(\nabla_+\Omega)_{+i} \Rightarrow &\qquad \omega_{+9i} = -\frac{1}{12}\phi_i{}^{j_1 j_2 j_3} F^{\mathbf{8}}_{+j_1 j_2 j_3} \\
(\nabla_+\Omega)_{i_1 i_2} \Rightarrow &\qquad F^{\mathbf{7}}_{+-i_1 i_2} = \frac{1}{24}\phi^{j_1 j_2 j_3}{}_{[i_1} F^{\mathbf{7}}_{i_2] j_1 j_2 j_3} \\
(\nabla_-\Omega)_{+i} \Rightarrow &\qquad \omega_{-9i} = 0 \\
(\nabla_{i_1}\Omega)_{+i_2} \Rightarrow &\qquad \omega_{i_1 i_2 9} = -\frac{1}{144}\delta_{i_1 i_2}\phi^{j_1 j_2 j_3 j_4} F^{\mathbf{1}}_{j_1 j_2 j_3 j_4} + \frac{1}{12}\phi_{(i_1}{}^{j_1 j_2 j_3} F^{\mathbf{1+35}}_{i_2) j_1 j_2 j_3} + \frac{1}{2}F_{+-i_1 i_2} \\
(\nabla_9\Omega)_{+i} \Rightarrow &\qquad \omega_{99i} = -\frac{1}{18}\phi_i{}^{j_1 j_2 j_3} F^{\mathbf{8}}_{9 j_1 j_2 j_3} - \frac{1}{3}F_{+-9i} \,.
\end{aligned} \tag{3.11}$$

We now turn to the conditions arising from derivatives of Σ. From (2.16) we obtain

$$\frac{1}{5!}\epsilon_{i_1 i_2 i_3}{}^{j_1 j_2 j_3 j_4 j_5}(d\phi)_{j_1 j_2 j_3 j_4 j_5} = -\frac{2}{3}F_{+-9j}\phi^j{}_{i_1 i_2 i_3} + \frac{2}{3}F_{i_1 i_2 i_3 9} - \frac{1}{2}F_{j_1 j_2 9 [i_1}\phi^{j_1 j_2}{}_{i_2 i_3]} \,. \tag{3.12}$$

In fact this equation fixes $\omega^{\mathbf{7}}_{i j_1 j_2} \equiv \frac{1}{4}(\omega_{i j_1 j_2} - \frac{1}{2}\phi_{j_1 j_2}{}^{k_1 k_2}\omega_{i k_1 k_2})$. To see this define

$$\psi_{i_1 i_2 i_3} \equiv \frac{1}{5!}\epsilon_{i_1 i_2 i_3}{}^{j_1 j_2 j_3 j_4 j_5}(d\phi)_{j_1 j_2 j_3 j_4 j_5} \tag{3.13}$$

then using (C.2) we deduce

$$\begin{aligned}
\psi_{i_1 i_2 i_3} &= -\phi^k{}_{i_1 i_2 i_3}\omega^j{}_{jk} + 3\phi_{j_1 j_2 [i_1 i_2}\omega^{j_1 j_2}{}_{i_3]} \\
&= -\phi^k{}_{i_1 i_2 i_3}\omega^{\mathbf{7}j}{}_{jk} + 3\phi_{j_1 j_2 [i_1 i_2}\omega^{\mathbf{7}j_1 j_2}{}_{i_3]} \,.
\end{aligned} \tag{3.14}$$

[4] In the most general supersymmetric geometries of $D = 6$ minimal supergravity there is always a null Killing vector which is also not hyper-surface orthogonal in general [4].

This expression can be inverted to give[5]

$$\omega^{\mathbf{7}}_{ij_1j_2} = \frac{1}{16}\psi_{ij_1j_2} + \frac{1}{48}\psi_{k_1k_2k_3}\delta_{i[j_1}\phi^{k_1k_2k_3}{}_{j_2]} - \frac{1}{32}\psi_{ik_1k_2}\phi^{k_1k_2}{}_{j_1j_2} + \\ + \frac{1}{16}\psi_{k_1k_2[j_1}\phi^{k_1k_2}{}_{j_2]i} \tag{3.15}$$

and hence in terms of the gauge field strength

$$\omega^{\mathbf{7}}_{ij_1j_2} = -\frac{1}{72}\delta_{i[j_1}\phi_{j_2]}{}^{k_1k_2k_3}F_{k_1k_2k_39} - \frac{1}{24}\phi_{j_1j_2}{}^{k_1k_2}F_{ik_1k_29} + \frac{1}{24}\phi_{i[j_1}{}^{k_1k_2}F_{j_2]k_1k_29} - \\ - \frac{1}{12}\delta_{i[j_1}F_{j_2]+-9} + \frac{1}{24}\phi_{ij_1j_2}{}^{k}F_{k+-9} + \frac{1}{12}F_{ij_1j_29}\,. \tag{3.16}$$

In later calculations it will be useful to note that the totally anti-symmetric part of $\omega^{\mathbf{7}}_{j_1j_2j_3}$ can be written as:

$$\omega^{\mathbf{7}}_{[j_1j_2j_3]} = -\frac{1}{16}\psi^{\mathbf{8}}_{j_1j_2j_3} + \frac{1}{12}\psi^{\mathbf{48}}_{j_1j_2j_3}\,. \tag{3.17}$$

This expression and (3.15) imply that $\omega^{\mathbf{7}}_{j_1j_2j_3}$ is fixed by the totally anti-symmetric part.

Finally from (2.13) we obtain

$$\begin{aligned} (\nabla_+\Sigma)_{+i_1i_2i_3i_4} &\Rightarrow & F^{\mathbf{7}}_{+9i_1i_2} &= 2\omega^{\mathbf{7}}_{+i_1i_2} \\ (\nabla_-\Sigma)_{+i_1i_2i_3i_4} &\Rightarrow & \omega^{\mathbf{7}}_{-ij} &= 0 \\ (\nabla_9\Sigma)_{+i_1i_2i_3i_4} &\Rightarrow & \omega^{\mathbf{7}}_{9i_1i_2} &= -\frac{1}{72}\phi^{j_1j_2j_3}{}_{i_1}F^{\mathbf{7}}_{i_2j_1j_2j_3} - \frac{1}{6}F^{\mathbf{7}}_{+-i_1i_2}\,. \end{aligned} \tag{3.18}$$

The conditions we have derived for the geometry and four-form to admit null Killing spinors are in fact sufficient, as we shall show in the next subsection. The careful reader will notice that the components $F^{\mathbf{48}}_{+i_1i_2i_3}$, $F^{\mathbf{21}}_{+9i_1i_2}$ and $F^{\mathbf{27}}_{i_1\ldots i_4}$ have not been constrained at all. The reason for this is, as we shall see, that these components of the field strength drop out of the Killing spinor equation. Note that a similar phenomenon was observed for timelike Killing spinors in [1].

3.2 Sufficiency

We would like to show that the conditions derived in the last sub-section are sufficient for the existence of null Killing spinors, satisfying the projections (2.7). Let us first derive some useful identities. Using the fact that

$$\frac{1}{4!}\phi_{i_1i_2i_3i_4}\Gamma^{i_1i_2i_3i_4}\epsilon = 14\epsilon \tag{3.19}$$

[5]We note in passing that this expression gives a formula for minus the intrinsic con-torsion of a general Spin(7)-structure in eight dimensions.

we obtain

$$
\begin{aligned}
\Gamma_i\epsilon &= \frac{1}{42}\phi_i{}^{j_1j_2j_3}\Gamma_{j_1j_2j_3}\epsilon \\
\Gamma_{i_1i_2}\epsilon &= -\frac{1}{6}\phi_{i_1i_2}{}^{j_1j_2}\Gamma_{j_1j_2}\epsilon \\
\Gamma_{i_1i_2i_3}\epsilon &= -\frac{1}{4}\phi_{[i_1i_2}{}^{j_1j_2}\Gamma_{i_3]j_1j_2}\epsilon = -\phi_{i_1i_2i_3}{}^{j}\Gamma_j\epsilon \\
\Gamma_{i_1i_2i_3i_4}\epsilon &= \frac{1}{4!}\epsilon_{i_1i_2i_3i_4}{}^{j_1j_2j_3j_4}\Gamma_{j_1j_2j_3j_4}\epsilon = \phi_{[i_1i_2i_3}{}^{j}\Gamma_{i_4]j}\epsilon + \phi_{i_1i_2i_3i_4}\epsilon \\
\Gamma_{i_1i_2i_3i_4i_5}\epsilon &= 5\phi_{[i_1i_2i_3i_4}\Gamma_{i_5]}\epsilon\,.
\end{aligned}
\tag{3.20}
$$

In particular we see that

$$
\begin{aligned}
\Gamma^{\mathbf{21}}_{i_1i_2}\epsilon &= 0 \\
\Gamma^{\mathbf{48}}_{i_1i_2i_3}\epsilon &= 0 \\
\Gamma^{\mathbf{27}}_{i_1i_2i_3i_4}\epsilon &= 0 \\
\Gamma^{\mathbf{35}}_{i_1i_2i_3i_4}\epsilon &= 0\,.
\end{aligned}
\tag{3.21}
$$

Now the Killing spinor equation is $\nabla_\alpha\epsilon + \frac{1}{288}M_\alpha\epsilon = 0$ where

$$
M_\alpha \equiv \Gamma_\alpha{}^{\nu_1\nu_2\nu_3\nu_4}F_{\nu_1\nu_2\nu_3\nu_4} - 8\Gamma^{\nu_1\nu_2\nu_3}F_{\alpha\nu_1\nu_2\nu_3}\,. \tag{3.22}
$$

Using the above identities, together with the constraints $\Gamma^9\epsilon = \epsilon$ and $\Gamma^+\epsilon = 0$, and the expressions for F presented in the last subsection, it is straightforward to show that

$$
\begin{aligned}
M_-\epsilon &= 0 \\
M_+\epsilon &= \Big[\Gamma^-\left(\phi^{i_1i_2i_3i_4}F_{i_1i_2i_3i_4} + 4\Gamma^q\phi^{i_1i_2i_3}{}_qF_{9i_1i_2i_3} + 48\Gamma^qF_{+-9q}\right) - \\
&\quad - 36\Gamma^{i_1i_2}F_{+9i_1i_2} + 12\Gamma^q\phi^{i_1i_2i_3}{}_qF_{+i_1i_2i_3}\Big]\epsilon \\
M_9\epsilon &= \Big[\Gamma^{ij}\left(F_{i_1i_2i_3[i}\phi^{i_1i_2i_3}{}_{j]} + 12F_{+-ij}\right) + 8\Gamma^j\left(\phi^{i_1i_2i_3}{}_jF_{9i_1i_2i_3} - 6F_{+-9j}\right) + \phi^{i_1i_2i_3i_4}F_{i_1i_2i_3i_4}\Big]\epsilon \\
M_i\epsilon &= \Big[72\Gamma^-\Gamma^jF_{-9ij} - 4\phi_i{}^{j_1j_2j_3}F_{9j_1j_2j_3} - 48F_{+-i9} + \\
&\quad + \Gamma^j\Big(\delta_{ij}\phi^{j_1j_2j_3j_4}F_{j_1j_2j_3j_4} - 4\phi_i{}^{k_1k_2k_3}F_{jk_1k_2k_3} - 12\phi_{ij}{}^{k_1k_2}F_{+-k_1k_2} + \\
&\qquad + 8\phi^{k_1k_2k_3}{}_jF_{ik_1k_2k_3} - 48F_{+-ij}\Big) + \Gamma^{j_1j_2} \times \\
&\quad \times \Big(3F_{9k_1k_2[j_1}\phi_{j_2]i}{}^{k_1k_2} - \delta_{i[j_1}\phi_{j_2]k_1k_2k_3}F_9{}^{k_1k_2k_3} + 24\delta_{i[j_1}F_{j_2]+-9} - 24F_{i9j_1j_2}\Big)\Big]\epsilon\,.
\end{aligned}
\tag{3.23}
$$

From these expressions and using (3.21) it is clear that $F^{\mathbf{48}}_{+i_1i_2i_3}$, $F^{\mathbf{21}}_{+9i_1i_2}$ and $F^{\mathbf{27}}_{i_1\ldots i_4}$ do not appear in $M_\mu\epsilon$ and hence these components are not fixed by the Killing spinor equation. Moreover, by making use of these expressions, and using $V_{ij}\Gamma^{ij}\epsilon = V^{\mathbf{7}}_{ij}\Gamma^{ij}\epsilon$, we find that

$$
\frac{1}{4}\omega_{\mu\alpha\beta}\Gamma^{\alpha\beta}\epsilon + \frac{1}{288}M_\mu\epsilon = 0 \tag{3.24}
$$

and hence the Killing spinor equation simplifies to

$$
d\epsilon = 0\,. \tag{3.25}
$$

Hence the Killing spinor is constant and constrained by (2.7).

3.3 Summary

We have derived the necessary and sufficient conditions on configurations admitting null Killing spinors. Here we shall summarise the results. Conceptually, it is clearest to separate the conditions into a set of restrictions on the spin connection, which are restrictions on the intrinsic torsion of the $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, and a set of conditions that determine the field strength in terms of the geometry. In the frame

$$ds^2 = 2e^+e^- + e^ie^i + e^9e^9\,, \tag{3.26}$$

where $i = 1, \ldots, 8$ and $\phi = \frac{1}{4!}\phi_{i_1i_2i_3i_4}e^{i_1i_2i_3i_4}$ is the Spin(7) invariant four-form given in (2.9), we have found the following constraints on the spin connection

$$\begin{aligned}
\omega_{(\alpha\beta)-} &= 0\\
\omega^{\mathbf{7}}_{[ij]-} &= 0\\
\omega_{i9-} &= 0\\
\omega_{-9i} &= 0\\
\omega^{\mathbf{7}}_{-ij} &= 0\\
\omega_{+-9} &= -\frac{1}{4}\omega^i{}_{i9}\\
\omega^{\mathbf{7}}_{9ij} &= -\omega^{\mathbf{7}}_{[ij]9}\\
\omega_{99i} - 6\omega_{i-+} &= -\frac{4}{3}\phi_i{}^{j_1j_2j_3}\omega^{\mathbf{7}}_{j_1j_2j_3}\,.
\end{aligned} \tag{3.27}$$

The bold-faced superscripts refer to Spin(7) representations. Note that the right hand side of the last term can also be written as $-\phi_i{}^{j_1j_2j_3}\omega_{j_1j_2j_3} - 2\omega^j{}_{ji}$.

Given a geometry satisfying the above restrictions the field strength is determined by,

$$\begin{aligned}
F_{+-9i} &= 2\omega_{i-+} - \omega_{99i}\\
F_{+-ij} &= 2\omega_{[ij]9}\\
F^{\mathbf{7}}_{+9ij} &= 2\omega^{\mathbf{7}}_{+ij}\\
F^{\mathbf{8}}_{+i_1i_2i_3} &= \frac{2}{7}\phi_{i_1i_2i_3}{}^j\omega_{+9j}\\
F^{\mathbf{7}}_{-9ij} &= 0\\
F^{\mathbf{21}}_{-9ij} &= 2\omega^{\mathbf{21}}_{ij-}\\
F_{-i_1i_2i_3} &= 0\\
F^{\mathbf{8}}_{9i_1i_2i_3} &= \frac{2}{7}\phi_{i_1i_2i_3}{}^j(\omega_{99j} + \omega_{j-+})\\
F^{\mathbf{48}}_{9i_1i_2i_3} &= -12(\omega^{\mathbf{7}}_{[i_1i_2i_3]})^{\mathbf{48}}\\
F^{\mathbf{1}}_{i_1i_2i_3i_4} &= \frac{3}{7}(\omega_{+9-})\phi_{i_1i_2i_3i_4}\\
F^{\mathbf{7}}_{i_1i_2i_3i_4} &= 2\phi_{[i_1i_2i_3}{}^j\omega^{\mathbf{7}}_{i_4]j\;9}\\
F^{\mathbf{35}}_{i_1i_2i_3i_4} &= 2\phi_{[i_1i_2i_3}{}^j\omega^{\mathbf{35}}_{i_4]j9}\,,
\end{aligned} \tag{3.28}$$

where $\omega^{\mathbf{35}}_{ij9} = \omega_{(ij)9} - \frac{1}{8}\delta_{ij}\omega^k{}_{k9}$ and $(\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{48}}$ is the **48** piece of the totally anti-symmetric part of $\omega^{\mathbf{7}}_{ijk}$. Note also that $\omega^{\mathbf{7}}_{ij9}$ denotes the **7** piece of $\omega_{[ij]9}$. The remaining components of the field strength, $F^{\mathbf{48}}_{+i_1i_2i_3}$, $F^{\mathbf{21}}_{+9i_1i_2}$ and $F^{\mathbf{27}}_{i_1..i_4}$, are undetermined by the Killing spinor equation as shown in the previous section, but are fixed by the Bianchi identity and gauge field equations, which we now discuss.

3.4 Conditions for supersymmetric solutions

It was shown in [1] that in order for a configuration (g, F) with a null Killing spinor to also solve the equations of motion of $D = 11$ supergravity, it is sufficient to just impose both the equation of motion and the Bianchi identity for F and in addition the $++$ component of Einstein's equations. Clearly these conditions will constrain the components of F not constrained by the Killing spinor equation alone.

4. Introducing co-ordinates

To introduce co-ordinates, note that locally, we can choose co-ordinates v, u, z so that the vector fields dual to our chosen frame are given by

$$\begin{aligned} e^+ &= \frac{\partial}{\partial v} \\ e^- &= \alpha_1\frac{\partial}{\partial v} + \alpha_2\frac{\partial}{\partial u} \\ e^9 &= \beta_1\frac{\partial}{\partial v} + \beta_2\frac{\partial}{\partial u} + \beta_3\frac{\partial}{\partial z} \end{aligned} \tag{4.1}$$

with $\alpha_2 \neq 0$, $\beta_3 \neq 0$. If the remaining co-ordinates are x^M, $M = 1, \ldots, 8$, then as a consequence of $i_{e^+}e^i = i_{e^-}e^i = i_{e^9}e^i = 0$ we obtain

$$e^i = e^i{}_M dx^M . \tag{4.2}$$

Inverting (4.1) we find that

$$\begin{aligned} e^+ &= \frac{1}{\alpha_2}du - \frac{\beta_2}{\alpha_2\beta_3}dz + \lambda \\ e^- &= -\frac{\alpha_1}{\alpha_2}du + dv + \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_3} - \frac{\beta_1}{\beta_3}\right)dz + \nu \\ e^9 &= \frac{1}{\beta_3}dz + \sigma , \end{aligned} \tag{4.3}$$

where $\lambda = \lambda_M dx^M$, $\nu = \nu_M dx^M$, $\sigma = \sigma_M dx^M$. By examining the $dudv$ and du^2 components of the metric, it is clear that as K is Killing, α_1 and α_2 do not depend on v. Furthermore, on examination of the $dvdz$, $dudz$ and dz^2 components of the metric, we also find that β_1, β_2 and β_3 must also be independent of v; and from the $dvdx^M$, $dudx^M$ and $dzdx^M$ components, it is clear that $\mathcal{L}_K\lambda = \mathcal{L}_K\nu = \mathcal{L}_K\sigma = 0$. The $dx^M dx^N$ components of the metric then imply that $\mathcal{L}_K(e^ie^i) = 0$; we shall find it convenient to refer to the 2-parameter family of 8-manifolds equipped with metric

$$ds_8{}^2 = e^ie^i \tag{4.4}$$

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as the base space B. Next note that from the differential constraints (3.27) we obtain

$$\mathcal{L}_K e^i = \rho^i e^+ + \chi_{ij} e^j \,, \tag{4.5}$$

where $\chi_{ij} = -\omega_{ij-} - \omega_{-ij}$ and $\rho^i = \omega_{+i-} - \omega_{-i+}$. Note in particular, that $\chi_{(ij)} = 0$ and $\chi^{\mathbf{7}} = 0$. However, we also have

$$0 = \mathcal{L}_K(e^i \otimes e^i) = \rho \otimes e^+ + e^+ \otimes \rho \,, \tag{4.6}$$

where $\rho \equiv \rho^i e^i$. Hence we must have $\rho^i = 0$, and

$$\mathcal{L}_K e^i = \chi_{ij} e^j \,. \tag{4.7}$$

Note that we can choose a basis of B, e'^i where $\mathcal{L}_K e'^i = 0$. To see this consider the metric on B, which we denote by h where

$$h_{MN} \equiv \delta_{ij} e^i{}_M(v,u,z,x) e^j{}_N(v,u,z,x) \,. \tag{4.8}$$

By the above reasoning, h_{MN} does not depend on v, and so on evaluating $e^i{}_M(v,u,z,x)$ at $v = 0$ we find that

$$h_{MN} \equiv \delta_{ij} (e')^i{}_M(u,z,x) (e')^j{}_N(u,z,x) \,, \tag{4.9}$$

where

$$(e')^i{}_M(u,z,x) \equiv e^i{}_M(0,u,z,x) \,. \tag{4.10}$$

It is clear that e' defines a basis of B for which $\mathcal{L}_K e'^i = 0$. In fact, the coefficients of ϕ are also constant in this basis. This is because $\mathcal{L}_K \phi = 0$ as a consequence of $\chi^{\mathbf{7}} = 0$. Hence the components $\phi_{M_1M_2M_3M_4}$ do not depend on v. However, we also have

$$\phi_{M_1M_2M_3M_4} = \phi_{i_1i_2i_3i_4} e^{i_1}{}_{M_1}(v,u,z,x) e^{i_2}{}_{M_2}(v,u,z,x) e^{i_3}{}_{M_3}(v,u,z,x) e^{i_4}{}_{M_4}(v,u,z,x) \,. \tag{4.11}$$

So, by the same reasoning as used above, on evaluating $e^i{}_M(v,u,z,x)$ at $v = 0$, we must have

$$\phi_{M_1M_2M_3M_4} = \phi_{i_1i_2i_3i_4} e'^{i_1}{}_{M_1}(u,z,x) e'^{i_2}{}_{M_2}(u,z,x) e'^{i_3}{}_{M_3}(u,z,x) e'^{i_4}{}_{M_4}(u,z,x) \tag{4.12}$$

which implies that the components of ϕ in the basis e'^i are identical to those in the basis e^i. Hence, without loss of generality we can drop the primes and work with a basis e^i for which *both* $\mathcal{L}_K e^i = 0$ *and* the components of ϕ are of the canonical form given in (2.9).

To continue we will introduce a more convenient notation:

$$\begin{aligned}
e^+ &= L^{-1}(du + Adz + \lambda) \\
e^- &= dv + \frac{1}{2}\mathcal{F} du + Bdz + \nu \\
e^9 &= C(dz + \sigma) \\
e^i &= e^i_M dx^M \,,
\end{aligned} \tag{4.13}$$

where the Lie-derivative of the functions $L, \mathcal{F}, A, B, C$ and the one-forms $\lambda, \nu, \sigma, e^i$ with respect to K all vanish i.e. they are all functions of u, x^M and z only.

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It is convenient to define some notation. For a q-form on the base manifold

$$\Theta = \frac{1}{q!}\Theta_{M_1\dots M_q} dx^{M_1} \wedge \cdots \wedge dx^{M_q} \tag{4.14}$$

satisfying $\mathcal{L}_K\Theta = 0$, we define the restricted exterior derivative

$$\tilde{d}\Theta \equiv \frac{1}{(q+1)!}(q+1)\frac{\partial}{\partial x^{M_1}}\Theta_{M_2\dots M_{q+1}} dx^{M_1} \wedge \dots \wedge dx^{M_{q+1}} \tag{4.15}$$

and denote the Lie derivative on such forms with respect to $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial u}$ by ∂_z and ∂_u respectively. We define

$$\mathcal{D}\Theta \equiv \tilde{d}\Theta + (A\sigma - \lambda) \wedge \partial_u\Theta - \sigma \wedge \partial_z\Theta \tag{4.16}$$

so that

$$d\Theta = \mathcal{D}\Theta + Le^+ \wedge \partial_u\Theta + C^{-1}e^9 \wedge (\partial_z\Theta - A\partial_u\Theta)\,. \tag{4.17}$$

We also define

$$\begin{aligned} M_{ij} &= \delta_{ik}(\partial_u e^k)_j \\ \Lambda_{ij} &= \delta_{ik}(\partial_z e^k)_j\,. \end{aligned} \tag{4.18}$$

In general, Λ and M have no symmetry properties. Using this notation, it is straightforward to compute the spin connection. All of the components of the spin connection are presented in appendix D.

In order to examine the restrictions on the eleven dimensional geometry imposed by the constraints in (3.27), first observe that the basis (4.13) contains a great deal of gauge freedom. In general, there is not a single gauge choice that simplifies all solutions, so it is convenient to allow some gauge freedom in the final form of the geometry. Nevertheless, to simplify the resulting formulae, we will work in a gauge with $A = 0$, which can be achieved by making a shift of the form $u \to u + f(u, z, x^M)$. Working in this gauge, (3.27) implies that

$$\begin{aligned} [\mathcal{D}\lambda]^{\mathbf{7}}_{ij} &= 0 \\ \partial_z\lambda &= 0 \\ [\mathcal{D}\log(CL^{-3}) - \partial_z\sigma - 3\partial_u\lambda]_i &= -\frac{1}{2\cdot 4!}[\mathcal{D}\phi]_{ij_1..j_4}\phi^{j_1..j_4} \\ \partial_z \log L &= \frac{1}{2}\Lambda^i{}_i \\ \Lambda^{\mathbf{7}}_{[ij]} &= 0\,. \end{aligned} \tag{4.19}$$

Note that the last two equations can be expressed in terms of the Spin(7) structure ϕ as,

$$\partial_z\phi = (\partial_z \log L)\phi + \Upsilon^{\mathbf{35}}\,, \tag{4.20}$$

where, denoting by $\Lambda^{\mathbf{35}}$ the traceless symmetric part, we defined

$$\Upsilon^{\mathbf{35}}_{i_1\dots i_4} \equiv -4\phi_{i[j_1j_2j_3}\Lambda^{\mathbf{35}i}{}_{j_4]}\,. \tag{4.21}$$

Using the terminology of [30], we recall that a Spin(7) structure satisfying (4.20) is called conformally anti-self-dual. Note that on making a conformal re-scaling of the base metric $ds_8^2 = L^{1/2}\hat{e}^i\hat{e}^i$, (4.20) becomes

$$\partial_z\hat{\phi} = \hat{\Upsilon}^{\mathbf{35}}\,, \tag{4.22}$$

where $\hat{\phi} = L^{-1}\phi$ and $\hat{\Upsilon}^{\mathbf{35}} = L^{-1}\Upsilon^{\mathbf{35}}$. Thus it makes sense to write the conditions in terms of these conformally rescaled variables. We do this in the following summary, where we also write out the four-form field strength.

4.1 Summary

We have shown that coordinates (u, v, z, x^M) can be chosen so that the metric takes the form

$$ds^2 = 2e^+e^- + e^ie^i + e^9e^9\,, \tag{4.23}$$

where

$$\begin{aligned}
e^+ &= L^{-1}(du+\lambda)\\
e^- &= dv + \frac{1}{2}\mathcal{F}du + Bdz + \nu\\
e^9 &= C(dz+\sigma)\\
e^i &= L^{1/4}\hat{e}^i_M dx^M\,.
\end{aligned} \tag{4.24}$$

The eight-dimensional base manifold with metric $\hat{e}^i\hat{e}^i$ has Cayley four-form $\hat{\phi}$ given by (2.9), (with ϕ replaced by $\hat{\phi}$ and e^i replaced by $\hat{e}^i$). In general all quantities can depend on the co-ordinates (u, z, x^M).

Supersymmetry implies that the following constraints must hold

$$\begin{aligned}
(\mathcal{D}\lambda)^{\mathbf{7}} &= 0\\
\partial_z\lambda &= 0\\
\Big[\mathcal{D}\log(CL^{1/2}) - \partial_z\sigma - 3\partial_u\lambda\Big]_i &= -\frac{1}{48}[\mathcal{D}\hat{\phi}]_{ij_1..j_4}\hat{\phi}^{j_1\ldots j_4}\\
\partial_z\hat{\phi} &= \hat{\Upsilon}^{\mathbf{35}}\,,
\end{aligned} \tag{4.25}$$

where all indices are evaluated with respect to the $\hat{e}^i$ basis and the boldface numbers denote Spin(7) irreps of forms taken with respect to the Spin(7) structure $\hat{\phi}$. The derivative $\mathcal{D}$ is defined in (4.16) (with $A = 0$).

In addition, it is straightforward to show that the 4-form F is given by

$$\begin{aligned}
F =\; & e^+\wedge e^-\wedge e^9\wedge(L^{-1}\mathcal{D}L - C^{-1}\mathcal{D}C + \partial_u\lambda + \partial_z\sigma) + Ce^+\wedge e^-\wedge\mathcal{D}\sigma +\\
&+ e^+\wedge e^9\wedge\left(-L^{3/2}\hat{M}_{[ij]}\hat{e}^i\wedge\hat{e}^j - \mathcal{D}\nu + \mathcal{D}B\wedge\sigma + \frac{1}{2}\mathcal{D}\mathcal{F}\wedge\lambda\right)^{\mathbf{7}} + L^{-1}e^-\wedge e^9\wedge\mathcal{D}\lambda -\\
&- \frac{1}{42}L^{1/2}C^{-1}\hat{\phi}_{i_i i_2 i_3}{}^j\left[\left(\partial_u B - \frac{1}{2}\partial_z\mathcal{F}\right)\lambda + \partial_z\nu - \tilde{d}B + LC^2\partial_u\sigma\right]_j e^+\wedge\hat{e}^{i_1}\wedge\hat{e}^{i_2}\wedge\hat{e}^{i_3} -\\
&- L^{1/2}e^9\wedge\hat{\star}_8\mathcal{D}\hat{\phi} + \frac{L^{1/2}}{6}\hat{\phi}_{i_1 i_2 i_3}{}^j\,[\partial_u\lambda]_j\, e^9\wedge\hat{e}^{i_1}\wedge\hat{e}^{i_2}\wedge\hat{e}^{i_3} + \frac{1}{2}LC^{-1}\partial_z\hat{\phi} +\\
&+ \frac{3}{14}C^{-1}\partial_z L\hat{\phi} - \frac{1}{24}L^{1/2}C\hat{\phi}^j{}_{[i_1 i_2 i_3}\mathcal{D}\sigma_{i_4]j}\hat{e}^{i_1}\wedge\hat{e}^{i_2}\wedge\hat{e}^{i_3}\wedge\hat{e}^{i_4} + F_{\text{unfixed}}\,,
\end{aligned} \tag{4.26}$$

where, again, all indices are evaluated in the $\hat{e}^i$ basis, $\hat{\star}_8$ denotes the Hodge dual with respect to the metric $\hat{e}^i\hat{e}^i$ and $\hat{M}_{ij} = \delta_{ik}(\partial_u \hat{e}^k)_j$. $F_{\rm unfixed}$ contains the components $F^{\bf 48}_{+i_1i_2i_3}$, $F^{\bf 21}_{+9i_1i_2}$ and $F^{\bf 27}_{i_1..i_4}$, which are undetermined by the Killing spinor equation.

5. Examples

In this section we consider some special examples of supersymmetric geometries. These provide some concrete insight into the $({\rm Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$-structures that we have shown supersymmetry dictates. We first consider the resolved membranes of [34], presenting a new generalisation involving the addition of a gravitational wave, followed by the basic fivebrane solution.

5.1 Membranes and their resolution

The elementary membrane and fivebrane solutions of $D = 11$ supergravity admit 16 Killing spinors. Some of these are timelike and the corresponding SU(5) structure was displayed in [1]. However some of the spinors are null so these solutions also fall in the null case that we are studying here. Let us focus first on the membrane. The metric and field strength for this solution are given by,

$$\begin{aligned} ds^2 &= H^{-2/3}[-dt^2 + (dx^\sharp)^2 + dz^2] + H^{1/3} ds^2(\mathbb{R}^8) \\ F &= dt \wedge dx^\sharp \wedge dz \wedge d(H^{-1}) \end{aligned} \tag{5.1}$$

where the gauge equations imply that H is a harmonic function on $\mathbb{R}^8$.

One can generalize this solution by replacing the space transverse to the membrane (which is $\mathbb{R}^8$ in (5.1)) by any Spin(7) holonomy manifold [33]. An additional generalisation leads to the "resolved membrane" solutions of [34]. To see how this latter solution is related to our construction of solutions of eleven dimensional supergravity, we introduce a null frame

$$\begin{aligned} e^+ &= \frac{H^{-2/3}}{\sqrt{2}}(-dt + dx^\sharp) \\ e^- &= \frac{1}{\sqrt{2}}(dt + dx^\sharp) \\ e^9 &= H^{-1/3} dz \\ e^i &= H^{1/6} \hat{e}^i \,, \end{aligned} \tag{5.2}$$

where $ds^2(M_8) = \hat{e}^i\hat{e}^i$ is a Spin(7) holonomy metric. Recall that this implies that $\hat{\phi}$ is closed, $\tilde{d}\hat{\phi} = 0$. Both H and $\hat{e}^i$ are independent of $t, x^\sharp, z$. On setting $v = \frac{1}{\sqrt{2}}(t + x^\sharp)$, $u = \frac{1}{\sqrt{2}}(-t + x^\sharp)$ it is clear that (5.2) corresponds to the null basis given in (4.24) with $L = H^{\frac{2}{3}}$, $A = \mathcal{F} = B = 0$, $\lambda = \nu = \sigma = 0$, $C = H^{-\frac{1}{3}}$ and $e^i = H^{\frac{1}{6}} \hat{e}^i{}_M dx^M$. It is then simple to check that the constraints required for supersymmetry (4.25) are satisfied.

The expression for the field strength given in (4.26) takes the form

$$F = H^{-1} e^+ \wedge e^- \wedge e^9 \wedge dH + \hat{F}^{\bf 27} \,, \tag{5.3}$$

where we have allowed for a piece, $\hat{F}^{\bf 27} = \frac{1}{4!}\hat{F}^{\bf 27}{}_{i_1i_2i_3i_4} \hat{e}^{i_1} \wedge \hat{e}^{i_2} \wedge \hat{e}^{i_3} \wedge \hat{e}^{i_4}$, in the **27** on M_8 that is not fixed by supersymmetry.

Imposing the gauge equations of motion we find that $\hat{F}^{\mathbf{27}}$ must be closed (and hence harmonic) while the equation for H becomes

$$\hat{\nabla}^2 H = -\frac{1}{2}|\hat{F}^{\mathbf{27}}|^2\,, \tag{5.4}$$

where $\hat{\nabla}^2$ is the laplacian on M_8 and the norm of $\hat{F}^{\mathbf{27}}$ is taken in the metric $ds^2(M_8)$. The $++$ component of the Einstein equations imposes no further restriction. Such "resolved membrane" solutions were constructed in [34] although there the issue of supersymmetry was not discussed and the internal component of the field strength was only constrained to be self-dual. The supersymmetry of such solutions was discussed in [36, 37]. The condition on the internal flux given in these papers is exactly the statement that it should belong to the **27** of Spin(7).

A generalization of the membrane solution preserving just 1/32 supersymmetry was constructed in [35]. The generalization involved replacing $e^- \to e^- + (1/2)\mathcal{F}du + \nu$ in (5.2) where $\mathcal{F}, \nu$ depend just on the coordinates on the Spin(7) manifold. This is a supersymmetric solution provided that $\mathcal{F}$ is harmonic and

$$d *_8 d\nu = 0 \tag{5.5}$$

on M_8. This was interpreted as adding a wave along the membrane although the wave "profile" $\mathcal{F}$ was smeared in the direction u.

The solutions of [34, 35] can be combined to yield a new, more general solution, by including both the term $\hat{F}^{\mathbf{27}}$ and $\mathcal{F}, \nu$ where now we allow $\mathcal{F} = \mathcal{F}(u, z, x^M)$ and maintain $\nu = \nu(x^M)$. In addition to $\hat{F}^{\mathbf{27}}$, we let F_{unfixed} in (4.26) also contain the piece: $F^{\mathbf{21}}_{+9ij} = -(d\nu)^{\mathbf{21}}_{ij}$. The gauge equations again imply that ν satisfies (5.5) and $\hat{F}^{\mathbf{27}}$ is harmonic on M_8 and H satisfies (5.4), while the $++$ component of the Einstein equations gives,

$$\hat{\nabla}^2\mathcal{F} + H\frac{\partial^2\mathcal{F}}{\partial z^2} = 0\,. \tag{5.6}$$

In general these solutions will preserve 1/32 supersymmetry. Notice that the dependence of $\mathcal{F}$ on u is not fixed. This is as expected since a supersymmetric wave is allowed to have an arbitrary profile. Note also that there is a special case when the Spin(7) manifold is a product of two hyper-Kähler manifolds and $\nu = 0$; for this case the resulting solutions are special cases of a class of solutions presented in [38].

5.2 The fivebrane

The metric for the basic fivebrane solution can be written as

$$ds^2 = H^{-1/3}[-dt^2 + (dx^\sharp)^2] + H^{2/3}dz^2 + ds_8^2\,, \tag{5.7}$$

where

$$ds_8^2 = H^{-1/3}\left[(dx^1)^2 + \cdots + (dx^4)^2\right] + H^{\frac{2}{3}}\left[(dx^5)^2 + \cdots + (dx^8)^2\right] \tag{5.8}$$

and $H = H(z, x^5, x^6, x^7, x^8)$. On setting $v = \frac{1}{\sqrt{2}}(t + x^\sharp)$, $u = \frac{1}{\sqrt{2}}(-t + x^\sharp)$ it is clear that (5.7) corresponds to the null basis given in (4.24) with $C = L = H^{\frac{1}{3}}$, $A = \mathcal{F} = B = 0$, $\lambda = \nu = \sigma = 0$, and we split the base indices on the 8-manifold via $e^i = \{e^a, e^p\}$ for

$a, b = 1, \ldots, 4$ and $p, q = 5, \ldots, 8$. The vielbein on the 8-dimensional base $\hat{e}^i$ is therefore

$$\begin{aligned}\hat{e}^a &= H^{-1/4} dx^a \\ \hat{e}^p &= H^{1/4} dx^p \,.\end{aligned} \tag{5.9}$$

$\hat{\phi}$ is given by

$$\begin{aligned}-\hat{\phi} = H^{-1} dx^{1234} + H dx^{5678} + (dx^{1256} + dx^{1278} + dx^{3456} + dx^{3478} + dx^{1357} - dx^{1368} - \\ - dx^{1458} - dx^{1467} - dx^{2358} - dx^{2367} - dx^{2457} + dx^{2468}) \,.\end{aligned} \tag{5.10}$$

It is then straightforward to show that the constraints given in (4.25) are satisfied. In addition it is straightforward to show that (4.26) gives

$$F = -dz \wedge \star_4 (\nabla_p H dx^p) + \frac{1}{14} H^{-1} \partial_z H \hat{\phi} + \frac{H^{-1}}{2} \partial_z H \hat{e}^{1234} - \frac{H^{-1}}{2} \partial_z H \hat{e}^{5678} + F_{\text{unfixed}} \,, \tag{5.11}$$

where $\nabla_p = \frac{\partial}{\partial x^p}$ and $\star_4$ denotes the Hodge dual on $\mathbb{R}^4$ equipped with metric

$$ds_4^2 = (dx^5)^2 + (dx^6)^2 + (dx^7)^2 + (dx^8)^2 \tag{5.12}$$

and positive orientation fixed with respect to $dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8$. However, unlike the case of the simple (non-resolved) M2-brane, in order to recover the standard expression for the components $F_{i_1 i_2 i_3 i_4}$, it is necessary to include a contribution from $F^{\mathbf{27}}{}_{i_1 i_2 i_3 i_4}$ which is not fixed by the supersymmetry. This term is given by

$$F^{\mathbf{27}} = -\frac{1}{14} H^{-1} \partial_z H \left(\hat{\phi} + 7 \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3 \wedge \hat{e}^4 + 7 \hat{e}^5 \wedge \hat{e}^6 \wedge \hat{e}^7 \wedge \hat{e}^8 \right) \,. \tag{5.13}$$

Hence the field strength is given by

$$F = - \star_5 dH \,, \tag{5.14}$$

where $\star_5$ denotes the Hodge dual on $\mathbb{R}^5$ equipped with metric

$$ds_5^2 = dz^2 + (dx^5)^2 + (dx^6)^2 + (dx^7)^2 + (dx^8)^2 \tag{5.15}$$

and positive orientation fixed with respect to $dz \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8$. The Bianchi identity implies that H is harmonic on $\mathbb{R}^5$. The field equations for the four-form and the $++$ component of the Einstein equations lead to no further conditions, and we see that we have recovered the fivebrane solution.

6. Configurations with vanishing flux

When the flux vanishes, the local form of the most general supersymmetric configuration was written down in [30]. It is interesting to recover this result from our more general results. One approach is to use the co-ordinates and frame introduced in section 4, set $F = 0$ in (4.26) and then analyse the resulting metric. However, we find it easier to obtain the result of [30] by introducing co-ordinates afresh, as we now explain.

If the flux vanishes, from (2.14) we note that K, Ω and Σ are closed. In particular, there exists (at least locally) functions u and v, such that as a 1-form

$$e^+ = du \tag{6.1}$$

and as a vector

$$e^+ = \frac{\partial}{\partial v} \,. \tag{6.2}$$

Furthermore, as $e^+ \wedge de^9 = 0$ as a consequence of $d\Omega = 0$, we note that there must exist functions z and P such that

$$e^9 = dz + Pdu \,. \tag{6.3}$$

Next consider the e^i, and e^-. In these co-ordinates, in general we have

$$e^i = e^i_M dx^M + X^i du + Y^i dz \tag{6.4}$$

and

$$e^- = dv + p_1 du + p_2 dz + \nu_2 \,. \tag{6.5}$$

By making a (generally u, z dependent) co-ordinate transformation of the x^M we can work in co-ordinates for which $Y^i = 0$. Next, observe that by making a basis rotation of the form given in (2.10) we can work in a basis for which $X^i = 0$ and $P = 0$. Hence, we have shown that if the flux vanishes, we can without loss of generality take the basis (4.13) with $L = C = 1$, $A = 0$, $\lambda = \sigma = 0$. By making a shift in v we can also set $B = 0$. Observe that closure of Σ implies that $\partial_z \phi = 0$ and hence in particular, $\partial_z(e^i e^i) = 0$. By the same reasoning which was used in section four to demonstrate that e^i could be chosen to be independent of v, we can, without loss of generality, choose a basis e^i for which $\partial_z e^i = 0$ (together with $\mathcal{L}_K e^i = 0$). Note also that $\tilde{d}\phi = 0$.

In fact we can also set $\nu = 0$. To see this note that from the vanishing of ω_{+9i} we must have $\partial_z \nu = 0$. Hence by making a basis transformation of the form given in (2.10) (with $\alpha = 0$, $Q = 1$ and $p_i = \nu_i$), we can remove the ν term from e^- at the expense of adding a $\nu^i du$ term to e^i. However, as ν^i has no z-dependence, we can remove this term by making a z-independent co-ordinate transformation of the x^M.

To summarize, when the flux vanishes, we can without loss of generality work in a null basis with

$$\begin{aligned} e^+ &= du \\ e^- &= dv + \frac{1}{2}\mathcal{F} du \\ e^9 &= dz \\ e^i &= e^i{}_M dx^M \end{aligned} \tag{6.6}$$

with $\mathcal{F} = \mathcal{F}(u, x, z)$ and $e^i{}_M = e^i{}_M(u, x)$. In this basis, $\tilde{d}\phi = 0$, so ϕ is covariantly constant with respect the the Levi-Civita connection on the base manifold. Although ϕ does not have any z-dependence, it does generically have a dependence on u. In particular, using $\omega^{\mathbf{7}}{}_{+ij} = 0$ it follows that $M^{\mathbf{7}}{}_{[ij]} = 0$ and hence

$$\partial_u \phi = \mathcal{T}\phi + \Psi^{\mathbf{35}} \,, \tag{6.7}$$

where $\mathcal{T} = (1/2)M_{ii}$ i.e. ϕ is conformally anti-self-dual.[6] Given this metric, $\mathcal{F}$ is fixed by the $++$ component of the Einstein equations. Hence we have recovered the result of [30].

7. Conclusions

In this paper we have completed the classification initiated in [1] of solutions of eleven dimensional supergravity preserving 1/32 of the supersymmetry. Just as in the case of simpler, lower-dimensional supergravities, this classification provides an interesting and promising tool for the generation of new solutions. In addition, we have shown how several previously known solutions, such as the zero-flux solution of [30], the fivebrane and the resolved membrane [34] can we written in our formalism. We generalised the solutions of [34, 35] by adding a gravitational wave to the resolved membrane and the resulting configuration preserves just 1/32 supersymmetry.

Supersymmetric solutions of most physical interest preserve more than one supersymmetry. Although such solutions are included in our classification, it is clear that the presence of more linearly independent Killing spinors imposes additional constraints on the geometry. Note that, using different techniques, the classification of maximally supersymmetric configurations, preserving all 32 supersymmetries, was carried out in [29]. It would therefore be interesting to generalize our construction to accommodate additional linearly independent Killing spinors. It might be possible, for example, to classify all geometries preserving exactly four supersymmetries, or perhaps all those preserving more than 1/2 of the supersymmetry.

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A. Conventions

We use the signature $(-,+,\cdots,+)$. $D=11$ co-ordinate indices will be denoted $\mu,\nu,\ldots$ while tangent space indices will be denoted by $\alpha,\beta,\ldots$. The $D=11$ spinors we will use are Majorana. The gamma matrices satisfy

$$\{\Gamma_\alpha, \Gamma_\beta\} = 2\eta_{\alpha\beta} \tag{A.1}$$

and can be taken to be real in the Majorana representation. They satisfy, in our conventions, $\Gamma_{0123456789\sharp} = \epsilon_{0123456789\sharp} = 1$. For any $M, N \in R(32)$ we can perform a Fierz

[6] It is interesting to compare this with a generic solution, with $F \neq 0$, where ϕ is not conformally anti-self dual with respect to ∂_u; rather, ϕ is conformally anti-self-dual with respect to ∂_z.

rearrangement using:

$$\begin{aligned}
M_a{}^b N_c{}^d = \frac{1}{32}\Big\{ & (NM)_a{}^d \delta_c{}^b + (N\Gamma^{\alpha_1}M)_a{}^d (\Gamma_{\alpha_1})_c{}^b - \\
& - \frac{1}{2!}(N\Gamma^{\alpha_1\alpha_2}M)_a{}^d (\Gamma_{\alpha_1\alpha_2})_c{}^b - \frac{1}{3!}(N\Gamma^{\alpha_1\alpha_2\alpha_3}M)_a{}^d (\Gamma_{\alpha_1\alpha_2\alpha_3})_c{}^b + \\
& + \frac{1}{4!}(N\Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4}M)_a{}^d (\Gamma_{\alpha_1\alpha_2\alpha_3\alpha_4})_c{}^b \\
& + \frac{1}{5!}(N\Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}M)_a{}^d (\Gamma_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5})_c{}^b \Big\}, \qquad \text{(A.2)}
\end{aligned}$$

where $a, b, c, d = 1, \ldots, 32$.

Given a Majorana spinor ϵ its conjugate is given by $\bar{\epsilon} = \epsilon^T C$, where C is the charge conjugation matrix in $D = 11$ and satisfies $C^T = -C$. In the Majorana representation we can choose $C = \Gamma_0$. An important property of gamma matrices in $D = 11$ is that the matrix $C\Gamma_{\alpha_1\alpha_2\ldots\alpha_p}$ is symmetric for $p = 1, 2, 5$ and antisymmetric for $p = 0, 3, 4$ (the cases $p > 5$ are related by duality to the above).

The Hodge star of a p-form ω is defined by

$$*\omega_{\mu_1\ldots\mu_{11-p}} = \frac{\sqrt{-g}}{p!}\epsilon_{\mu_1\ldots\mu_{11-p}}{}^{\nu_1\ldots\nu_p}\omega_{\nu_1\ldots\nu_p} \qquad \text{(A.3)}$$

and the square of a p-form via

$$\omega^2 = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\omega^{\mu_1\ldots\mu_p} \qquad \text{(A.4)}$$

unless otherwise stated.

B. Algebraic relations of $D = 11$ spinors

Here we analyse the algebraic structure of the differential forms K, Ω, Σ defined in (2.4) using Fierz identities. This provides an alternative derivation of (2.5)-(2.9) which relied on some results of [30].

Using Fierz identities one finds:

$$\begin{aligned}
\Sigma^2 &= -6K^2 \\
\Omega^2 &= -5K^2 && \text{(B.1)} \\
\Omega_{\mu_1}{}^{\sigma_1}\Omega_{\sigma_1}{}^{\nu_1} &= -K_{\mu_1}K^{\nu_1} + \delta_{\mu_1}{}^{\nu_1}K^2 && \text{(B.2)} \\
\frac{1}{4!}\Sigma_{\mu_1}{}^{\sigma_1\sigma_2\sigma_3\sigma_4}\Sigma_{\sigma_1\sigma_2\sigma_3\sigma_4}{}^{\nu_1} &= 14K_{\mu_1}K^{\nu_1} - 4\delta_{\mu_1}{}^{\nu_1}K^2 && \text{(B.3)} \\
i_K\Omega &= 0 && \text{(B.4)} \\
i_K\Sigma &= \frac{1}{2}\Omega\wedge\Omega && \text{(B.5)} \\
K^\sigma(*\Sigma)_{\sigma\nu_1\nu_2\nu_3\nu_4\nu_5} &= \Omega_{\nu_1}{}^\sigma\Sigma_{\sigma\nu_2\nu_3\nu_4\nu_5} - 12\eta_{\nu_1[\nu_2}K_{\nu_3}\Omega_{\nu_4\nu_5]} && \text{(B.6)} \\
K^2\Omega\wedge\Sigma &= \frac{1}{2}K\wedge\Omega\wedge\Omega\wedge\Omega && \text{(B.7)} \\
\Omega_{\nu_1}{}^\rho \star \Sigma_{\rho\nu_2\nu_3\nu_4\nu_5\nu_6} &= -5\Sigma_{\nu_1[\nu_2\nu_3\nu_4\nu_5}K_{\nu_6]} + 5\eta_{\nu_1[\nu_2}(i_K\Sigma)_{\nu_3\nu_4\nu_5\nu_6]}\,. && \text{(B.8)}
\end{aligned}$$

Note that equations (B.1)–(B.7) appeared previously in [1] and that (B.6) corrects equation (2.14) of that reference.

These are by no means exhaustive, though they are in fact sufficient to deduce the algebraic structures in both timelike and null cases. In particular, (B.2) implies that K cannot be spacelike. To see this, note first that in a neighbourhood in which ϵ is non-vanishing, $K_0 = -\epsilon^T \epsilon \neq 0$. Without loss of generality, $K = -(\epsilon^T \epsilon) e^0 + m e^\sharp$. As $i_K \Omega = 0$ we must have

$$\left(\epsilon^T \epsilon\right) \Omega_{0\alpha} + m \Omega_{\sharp\alpha} = 0 \,. \tag{B.9}$$

In particular, we find $\Omega_{0\sharp} = 0$ and $\Omega_{0P} = -(m/\epsilon^T \epsilon)\Omega_{\sharp P}$, where $P, Q = 1, \dots, 9$. Then upon setting $\mu_1 = \nu_1 = \sharp$ in (B.2) we find that

$$\Omega_{\sharp P} \Omega_{\sharp}{}^{P} = m^2 - K^2 = (\epsilon^T \epsilon)^2 \,. \tag{B.10}$$

But setting $\mu_1 = P$, $\nu_1 = Q$ in (B.2) we see that

$$\delta_{PQ}(m^2 - (\epsilon^T \epsilon)^2) = (\epsilon^T \epsilon)^{-2}(m^2 - (\epsilon^T \epsilon)^2)\Omega_{P\sharp}\Omega_{Q\sharp} - \Omega_P{}^L \Omega_{QL} \,. \tag{B.11}$$

Contracting with δ^{PQ} we obtain

$$8\left(m^2 - \left(\epsilon^T \epsilon\right)^2\right) = -\Omega_{PQ}\Omega^{PQ} \,. \tag{B.12}$$

This implies that $m^2 \leq (\epsilon^T \epsilon)^2$, so K must be timelike or null.

The case when K is timelike has been examined in detail in [1]. Here we shall concentrate on the case when K is null. It is therefore convenient to work in a null basis

$$ds^2 = 2e^+ e^- + e^P e^P \tag{B.13}$$

with $K = e^+$. To proceed, note that $i_K \Omega = 0$ implies that

$$\Omega = e^+ \wedge V + \frac{1}{2}\Omega_{PQ} e^P \wedge e^Q \,, \tag{B.14}$$

where $V = V_P e^P$. However, as $\Omega^2 = 0$ it is straightforward to see that $\Omega_{PQ} = 0$, so

$$\Omega = e^+ \wedge V \,. \tag{B.15}$$

Setting $\mu_1 = +$, $\nu_1 = -$ in (B.2) we also find that $V^2 = 1$. Note that (B.15) implies that $\Omega \wedge \Omega = 0$ and hence from (B.5) we find that $i_K \Sigma = 0$, hence

$$\Sigma = e^+ \wedge \phi + \frac{1}{5!}\Sigma_{P_1P_2P_3P_4P_5} e^{P_1P_2P_3P_4P_5} \,, \tag{B.16}$$

where $\phi = \frac{1}{4!}\phi_{P_1P_2P_3P_4} e^{P_1P_2P_3P_4}$. However $\Sigma^2 = 0$, so $\Sigma_{P_1P_2P_3P_4P_5} = 0$, and hence

$$\Sigma = e^+ \wedge \phi \,. \tag{B.17}$$

In addition, from (B.6) we note that $\Omega_{\nu_1}{}^{\sigma}\Sigma_{\sigma\nu_2\nu_3\nu_4\nu_5} = 0$ as $i_K \star \Sigma = 0$ and $K \wedge \Omega = 0$. Setting $\nu_1 = \nu_2 = +$ we find that

$$i_V \phi = 0 \,. \tag{B.18}$$

Hence it is convenient to make an 8+1 split $e^P = \{e^i, e^9\}$ for $i, j = 1, \dots, 8$ with $V = e^9$ and $\phi = \frac{1}{4!}\phi_{i_1i_2i_3i_4} e^{i_1i_2i_3i_4}$. In addition, setting $\nu_1 = \nu_2 = +$ in (B.8) we note that ϕ is a self dual 4-form on the 8-manifold equipped with metric $\delta_{ij} e^i e^j$, where we take $\epsilon_{+-123456789} = -1$ with positive orientation on the 8-manifold given by $\epsilon_{12345678} = 1$.

To proceed, we work in a particular basis in which $K = -\epsilon^T \epsilon (e^0 + e^\sharp)$. By examining the expressions for K and V using the representation of Cliff$(1,10)$ presented below, we see that $\epsilon^a = 0$ for $a = 9, \ldots, 32$, or equivalently

$$\Gamma_9 \epsilon = \epsilon \tag{B.19}$$

and

$$(\Gamma_0 - \Gamma_\sharp)\epsilon = 0\,. \tag{B.20}$$

Moreover, a direct examination of the components of ϕ yields the identity

$$\phi^{i_1 i_2 i_3 j} \phi_{q_1 q_2 q_3 j} = 6\delta^{i_1 i_2 i_3}_{q_1 q_2 q_3} - 9\phi^{[i_1 i_2}{}_{[q_1 q_2}\delta^{i_3]}_{q_3]}\,. \tag{B.21}$$

In particular, we find that

$$\phi_{i_1 i_2 i_3 j} \phi_{i_1 i_2 i_3}{}^j = 1 \tag{B.22}$$

for distinct fixed i_1, i_2, i_3.

It appears that there are eight degrees of freedom in the spinor ϵ. In fact there is only one degree of freedom, and a basis $\{e^i\}$ can be chosen in which $-\phi$ takes the canonical form of the Cayley 4-form. To see this we shall concentrate on the components ϕ_{146i}, ϕ_{145i} and ϕ_{168i}. Observe from (B.22) that $\phi_{146i}\phi_{146}{}^i = 1$. Hence by rotating in the 2,3,5,7,8 directions we can arrange without loss of generality for $\phi_{1467} = 1$ and $\phi_{1462} = \phi_{1463} = \phi_{1465} = \phi_{1468} = 0$. By inspecting the expression for ϕ_{1467} in terms of components of the spinor, it is apparent that $\phi_{1467} = 1$ implies that $\epsilon^2 = \epsilon^5 = \epsilon^7 = \epsilon^8 = 0$. Next consider ϕ_{145i}; again we have $\phi_{145i}\phi_{145}{}^i = 1$. In addition, the only non-vanishing components of ϕ_{145i} are ϕ_{1452}, ϕ_{1453} and ϕ_{1458}. Hence, by rotating in the 2,3,8 directions we can set without loss of generality $\phi_{1458} = 1$ and $\phi_{1452} = \phi_{1453} = 0$. Note that such a rotation will not change the values of ϕ_{146i}. Moreover, $\phi_{1458} = 1$ implies that $\epsilon^6 = -\epsilon^3$ and $\epsilon^4 = \epsilon^1$. Lastly consider ϕ_{168i}; once more, from (B.22) we have $\phi_{168i}\phi_{168}{}^i = 1$. In addition, the only non-vanishing components of ϕ_{168i} are ϕ_{1682} and ϕ_{1683}. Hence, by rotating in the 2,3 directions we can set without loss of generality $\phi_{1683} = 1$, $\phi_{1682} = 0$. Such a rotation leaves unaltered the values of ϕ_{146i} and ϕ_{145i}, and $\phi_{1683} = 1$ implies that $\epsilon^3 = -\epsilon^1$.

To summarize, in this basis, we find that

$$\begin{aligned} -\phi = {} & e^{1234} + e^{1256} + e^{1278} + e^{3456} + e^{3478} + e^{5678} + \\ & + e^{1357} - e^{1368} - e^{1458} - e^{1467} - e^{2358} - e^{2367} - e^{2457} + e^{2468}\,, \end{aligned} \tag{B.23}$$

and the only non-vanishing components of the spinor ϵ are $\epsilon^1 = -\epsilon^3 = \epsilon^4 = \epsilon^6$. This corresponds to imposing the projections

$$\Gamma_{1234}\epsilon = \Gamma_{3456}\epsilon = \Gamma_{5678}\epsilon = \Gamma_{1357}\epsilon = -\epsilon\,. \tag{B.24}$$

In summary, we see that we have rederived equations (2.5)–(2.9).

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B.1 An explicit representation of Cliff(10,1)

In order to compute some of the Fierz identities and algebraic relations satisfied by the various bi-linears, it is useful to have an explicit representation for Cliff(10,1). We recall the representation given in [39]. In particular, let L_i denote left multiplication by the imaginary octonions on the octonions, for $i = 1, \ldots, 7$. Explicitly, if e_i for $i = 1, \ldots, 7$ denote the imaginary unit octonions, then we take

$$e_i.e_j = -\delta_{ij} + c_{ijk} e_k \,, \tag{B.25}$$

where c_{ijk} is totally skew and has non-vanishing components fixed (up to permutation of indices) by

$$c_{124} = c_{137} = c_{156} = c_{235} = c_{267} = c_{346} = c_{457} = 1 \,. \tag{B.26}$$

Then it is straightforward to construct the representation of Cliff(8,0) by defining the following 16×16 real block matrices

$$\begin{aligned} \hat{\Gamma}_i &= \begin{pmatrix} 0 & L_i \\ L_i & 0 \end{pmatrix} \\ \hat{\Gamma}_8 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{B.27}$$

for $i = 1, \ldots, 7$. The representation of Cliff(10,1) is then obtained by defining the following 32×32 real block matrices

$$\begin{aligned} \Gamma_i &= \begin{pmatrix} 0 & -\hat{\Gamma}_i \\ \hat{\Gamma}_i & 0 \end{pmatrix} \\ \Gamma_9 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \Gamma_\sharp &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{B.28}$$

for $i = 1, \ldots, 8$ and

$$\Gamma_0 = -\Gamma_{123456789\sharp} \,. \tag{B.29}$$

C. Spin(7) Identities

The Spin(7) 4-form ϕ satisfies the following identities, which, as far as we know, are new:

$$42\phi_{[i_1 i_2}{}^{[j_1 j_2}\delta^{j_3 j_4]}_{i_3 i_4]} + \phi_{i_1 i_2 i_3 i_4}\phi^{j_1 j_2 j_3 j_4} - 3\phi_{[i_1 i_2}{}^{[j_1 j_2}\phi^{j_3 j_4]}{}_{i_3 i_4]} + 2\phi^{[j_1}{}_{[i_1 i_2 i_3}\phi_{i_4]}{}^{j_2 j_3 j_4]} = 0 \tag{C.1}$$

and

$$\frac{1}{4!}\epsilon_{i_1 i_2 i_3 i_4}{}^{j_1 j_2 j_3 j_4} = \frac{1}{168}\phi_{i_1 i_2 i_3 i_4}\phi^{j_1 j_2 j_3 j_4} + \frac{3}{28}\phi_{[i_1 i_2}{}^{[j_1 j_2}\phi^{j_3 j_4]}{}_{i_3 i_4]} + \frac{2}{21}\phi^{[j_1}{}_{[i_1 i_2 i_3}\phi_{i_4]}{}^{j_2 j_3 j_4]} \,. \tag{C.2}$$

We also have the well known identities

$$\begin{aligned} \phi^{i_1 i_2 i_3 k}\phi_{j_1 j_2 j_3 k} &= 6\delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} - 9\phi^{[i_1 i_2}{}_{[j_1 j_2}\delta^{i_3]}_{j_3]} \\ \phi^{i_1 i_2 k_1 k_2}\phi_{j_1 j_2 k_1 k_2} &= 12\delta^{i_1 i_2}_{j_1 j_2} - 4\phi^{i_1 i_2}{}_{j_1 j_2} \\ \phi^{i k_1 k_2 k_3}\phi_{j k_1 k_2 k3} &= 42\delta^i_j \,. \end{aligned} \tag{C.3}$$

Given these identities one can show that

$$
\begin{aligned}
\phi^k{}_{[i_1 i_2 i_3}\alpha^{\mathbf{21}}_{i_4]k} &= 0 \\
\phi_{[i_1 i_2}{}^{j_1 j_2}\alpha_{i_3]j_1 j_2} &= \left(-4\alpha^{\mathbf{8}} + \frac{2}{3}\alpha^{\mathbf{48}}\right)_{i_1 i_2 i_3} \\
\phi_{[i_1 i_2}{}^{j_1 j_2}\alpha_{i_3 i_4]j_1 j_2} &= \left(-4\alpha^{\mathbf{1}} - 2\alpha^{\mathbf{7}} + \frac{2}{3}\alpha^{\mathbf{27}}\right)_{i_1 i_2 i_3 i_4} \\
\phi_{[i_1 i_2}{}^{j_1 j_2}\phi_{i_3 i_4]}{}^{j_3 j_4}\alpha_{j_1 j_2 j_3 j_4} &= \left(28\alpha^{\mathbf{1}} - 12\alpha^{\mathbf{7}} + \frac{28}{3}\alpha^{\mathbf{27}} - 4\alpha^{\mathbf{35}}\right)_{i_1 i_2 i_3 i_4} \\
\phi_{[i_1 i_2 i_3}{}^{j_1}\phi_{i_4]}{}^{j_2 j_3 j_4}\alpha_{j_1 j_2 j_3 j_4} &= \left(42\alpha^{\mathbf{1}} - 24\alpha^{\mathbf{7}} + 6\alpha^{\mathbf{35}}\right)_{i_1 i_2 i_3 i_4} \\
\phi^{j_1 j_2 j_3}{}_{i_1}\alpha^{\mathbf{7}}_{i_2 j_1 j_2 j_3} &= \phi^{j_1 j_2 j_3}{}_{[i_1}\alpha^{\mathbf{7}}_{i_2]j_1 j_2 j_3}\,.
\end{aligned}
\tag{C.4}
$$

D. Spin connection components

Using the definitions for $\tilde{d}$ and $\mathcal{D}$ in (4.15) and (4.16), note that

$$
\begin{aligned}
de^+ &= L^{-1}(\mathcal{D}\lambda - (\mathcal{D}A)\wedge\sigma) + e^+ \wedge (L^{-1}\mathcal{D}L + \partial_u\lambda - \partial_u A\sigma) + \\
&\quad + e^9 \wedge [(LC)^{-1}(\partial_z\lambda - A\partial_u\lambda + (\partial_u A)\lambda - \tilde{d}A)] + \\
&\quad + LC^{-1}(\partial_z L - A\partial_u L + L\partial_u A)e^+\wedge e^9 \\
de^- &= \left(\frac{1}{2}\mathcal{D}\mathcal{F}\wedge(A\sigma - \lambda) + \mathcal{D}\nu - (\mathcal{D}B)\wedge\sigma\right) + \\
&\quad + e^+ \wedge \left[L\left(-\frac{1}{2}\tilde{d}\mathcal{F} + (\frac{1}{2}\partial_z\mathcal{F} - \partial_u B)\sigma + \partial_u\nu\right)\right] + \\
&\quad + e^9 \wedge \left[C^{-1}\left(\frac{1}{2}A\tilde{d}\mathcal{F} + (\partial_u B - \frac{1}{2}\partial_z\mathcal{F})\lambda - \tilde{d}B + \partial_z\nu - A\partial_u\nu\right)\right] + \\
&\quad + LC^{-1}\left(-\frac{1}{2}\partial_z\mathcal{F} + \partial_u B\right)e^+\wedge e^9 \\
de^9 &= C\mathcal{D}\sigma + e^+\wedge(LC\partial_u\sigma) + e^9\wedge\left(-C^{-1}\mathcal{D}C + \partial_z\sigma - A\partial_u\sigma\right) + \\
&\quad + LC^{-1}(\partial_u C)e^+\wedge e^9 \\
de^i &= \mathcal{D}e^i + e^+\wedge\left(L\partial_u e^i\right) + e^9\wedge\left[C^{-1}\left(\partial_z e^i - A\partial_u e^i\right)\right].
\end{aligned}
\tag{D.1}
$$

Using these expressions, the following non-vanishing components of the spin connection are obtained:

$$
\begin{aligned}
\omega_{9-+} &= \frac{C^{-1}}{2}(A\partial_u \log L^{-1} + \partial_u A - \partial_z \log L^{-1}) \\
\omega_{i-+} &= -\frac{1}{2}(\mathcal{D}\log L^{-1} + \partial_u A\ \sigma - \partial_u\lambda)_i
\end{aligned}
\tag{D.2}
$$

$$
\begin{aligned}
\omega_{+-9} &= -\frac{C^{-1}}{2}(A\partial_u \log L^{-1} + \partial_u A - \partial_z \log L^{-1}) \\
\omega_{i-9} &= -\frac{(LC)^{-1}}{2}(\tilde{d}A + A\partial_u\lambda - \partial_z\lambda - \partial_u A\ \lambda)_i
\end{aligned}
\tag{D.3}
$$

$$
\omega_{+-i} = \frac{1}{2}(\mathcal{D}\log L^{-1} + \partial_u A\ \sigma - \partial_u\lambda)_i
$$

$$
\begin{aligned}
\omega_{9-i} &= \frac{(LC)^{-1}}{2}(\tilde{d}A + A\partial_u\lambda - \partial_z\lambda - \partial_u A\ \lambda)_i \\
\omega_{j-i} &= \frac{L^{-1}}{2}(\mathcal{D}\lambda - \mathcal{D}A\wedge\sigma)_{ij}
\end{aligned}
\tag{D.4}
$$

$$
\begin{aligned}
\omega_{-+i} &= \frac{1}{2}(\mathcal{D}\log L^{-1} + \partial_u A\ \sigma - \partial_u\lambda)_i \\
\omega_{++i} &= L\left[\left(\partial_u B - \frac{1}{2}\partial_z\mathcal{F}\right)\sigma - \partial_u\nu + \frac{1}{2}\tilde{d}\mathcal{F}\right]_i \\
\omega_{9+i} &= -\frac{C^{-1}}{2}\left[\left(\partial_u B - \frac{1}{2}\partial_z\mathcal{F}\right)\lambda + \partial_z\nu - A\partial_u\nu + \frac{1}{2}A\tilde{d}\mathcal{F} - \tilde{d}B\right]_i - \frac{LC}{2}(\partial_u\sigma)_i \\
\omega_{j+i} &= -LM_{(ij)} + \frac{1}{2}\left[\mathcal{D}\nu - \mathcal{D}B\wedge\sigma + \frac{1}{2}\mathcal{D}\mathcal{F}\wedge(A\sigma - \lambda)\right]_{ij}
\end{aligned}
\tag{D.5}
$$

$$
\begin{aligned}
\omega_{++9} &= -LC^{-1}\left(\partial_u B - \frac{1}{2}\partial_z\mathcal{F}\right) \\
\omega_{-+9} &= -\frac{C^{-1}}{2}(A\partial_u\log L^{-1} + \partial_u A - \partial_z\log L^{-1}) \\
\omega_{9+9} &= -L\partial_u\log C \\
\omega_{i+9} &= \frac{C^{-1}}{2}\left[\left(\partial_u B - \frac{1}{2}\partial_z\mathcal{F}\right)\lambda + \partial_z\nu - A\partial_u\nu + \frac{1}{2}A\tilde{d}\mathcal{F} - \tilde{d}B\right]_i - \frac{LC}{2}(\partial_u\sigma)_i
\end{aligned}
\tag{D.6}
$$

$$
\begin{aligned}
\omega_{-9i} &= \frac{(LC)^{-1}}{2}(\tilde{d}A + A\partial_u\lambda - \partial_z\lambda - \partial_u A\ \lambda)_i \\
\omega_{+9i} &= -\frac{C^{-1}}{2}\left[\left(\partial_u B - \frac{1}{2}\partial_z\mathcal{F}\right)\lambda + \partial_z\nu - A\partial_u\nu + \frac{1}{2}A\tilde{d}\mathcal{F} - \tilde{d}B\right]_i - \frac{LC}{2}(\partial_u\sigma)_i \\
\omega_{99i} &= (\mathcal{D}\log C + A\partial_u\sigma - \partial_z\sigma)_i \\
\omega_{j9i} &= -C^{-1}\Lambda_{(ij)} + C^{-1}AM_{(ij)} + \frac{C}{2}(\mathcal{D}\sigma)_{ij}
\end{aligned}
\tag{D.7}
$$

$$
\begin{aligned}
\omega_{+ij} &= -LM_{[ij]} - \frac{1}{2}\left[\mathcal{D}\nu - \mathcal{D}B\wedge\sigma + \frac{1}{2}\mathcal{D}\mathcal{F}\wedge(A\sigma - \lambda)\right]_{ij} \\
\omega_{-ij} &= -\frac{L^{-1}}{2}(\mathcal{D}\lambda - \mathcal{D}A\wedge\sigma)_{ij} \\
\omega_{9ij} &= -C^{-1}\Lambda_{[ij]} + C^{-1}AM_{[ij]} - \frac{C}{2}(\mathcal{D}\sigma)_{ij} \\
\omega_{kij} &= \tilde{\omega}_{kij} + \sigma_{[i}\Lambda_{|k|j]} + \sigma_k\Lambda_{[ij]} + \sigma_{[i}\Lambda_{j]k} - \\
&\quad - (A\sigma - \lambda)_{[i}M_{|k|j]} - (A\sigma - \lambda)_k M_{[ij]} - (A\sigma - \lambda)_{[i}M_{j]k}\,,
\end{aligned}
\tag{D.8}
$$

where $\tilde{\omega}$ denotes the spin connection of the base space.

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